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Advances in Arithmetic: Introducing the Errorless Calculator

Shane Mathew Palmquist
Western Kentucky University
shane.palmquist@wku.edu

Timothy J. Penn
Western Kentucky University
timothy.penn769@topper.wku.edu

Abstract

The topic of division by zero has been debated for centuries. From grade school to post-secondary education, our youth have repeatedly been taught "You can't divide by zero.", "Dividing by zero is impossible.", "It's undefined." Rather than presenting another scholarly work supporting the concept of division by zero of having a numerical solution, the goal of this paper was to develop a new arithmetic errorless calculator software tool that eliminates mathematical paradoxes. These paradoxes include arithmetic that results in division by zero as well as indeterminant forms such as zero divided by zero. To do this, a new number system will be briefly introduced and is referred to as omnifinite numbers. Using this proposed number system, the resulting calculator tool was found to be simple to use, intuitive, and fun. In addition, all numeric outputs performed by the calculator tool are arithmetically error free. The presentation of this work at the conference shall give attendees in real time an opportunity to use the errorless calculator tool hands-on and ask questions. This work serves as a manual of instruction in support of using and understanding this new technological software tool. To the authors' knowledge, this is the world's first arithmetic errorless calculator. It is the hope of the authors that this work also serves as the start of a bridge to the future of greater mathematical understanding and knowledge which will lead to improvements in technology across all fields, and this all begins with the building blocks of fundamental mathematics, specifically numbers.

Introduction to Omnifinite Numbers

This paper introduces the concept of omnifinite numbers, which is a new but similar system of numbers as compared to the hyperreals. A detailed discussion regarding the properties of omnifinite numbers is beyond the scope of this work and is not needed for understanding the basics of these numbers and how the errorless calculator software tool works and functions. In general, omnifinites are a more robust and complete number system than other number systems commonly used today such as the reals or hyperreals. As mentioned, omnifinite numbers are very similar to hyperreals in that numbers may be finite or nonfinite. Nonfinite numbers may be infinitesimal, infinite, or a combination. In short, the omnifinites, like the hyperreals, allow for very small, normal, and very large numbers that the reals are unable to adequately describe since they are finite. A brief discussion and overview of some of the important aspects and principles of omnifinites is presented in this section. The next section will present some of the properties of omnifinite numbers to help better understand how zero and differing infinities and infinitesimals work within the context of mathematical arithmetic. Of importance throughout this paper is basic arithmetic which is what led to the development of the errorless calculator software tool.

For this work in developing and creating the errorless calculator, the authors introduce a new set of numbers referred to as omnifinites. Generally, in mathematics, a number is an arithmetic value used to represent a quantity. This definition implicitly implies a concept of size as well as order, but not explicitly. For omnifinites, this definition is used as well. The Greeks, in defining and describing a number, the concept of "The part of a number is less than the whole." was foundational (Sergeyev, 2013). To the Greeks, this concept was a governing principle to all numbers, and for the Greeks, this meant all real numbers, finite numbers. Omnifinite numbers, which include all finite and all nonfinite numbers, share this fundamental principle regardless of the size of the number. This is important distinction which will be further explored and explained in an upcoming work on omnifinite set theory as compared to Prof. Georg Cantor's set theory (Cantor, 1890).

Omninfinites, $\mathbb{O}$, may be easily compared to other more familiar number systems like the reals, $\mathbb{R}$, and the hyperreals, $^*\mathbb{R}$, as shown in Figure 1. The line representing the reals is dashed in Figure 1 since all infinitesimals have been removed or compacted out of the number system. Unlike the reals and hyperreals, omnifinites are a closed system of numbers. The number line systems for the reals and hyperreals as shown in Figure 1 have arrows at both ends of the line segment indicating that they continue in both directions in a finite and nonfinite directions, respectively. The omnifinite number line starts with a number and ends at another number. There are no graphical arrows associated with the omnifinite number system. Thus, the omnifinite number system is closed and not open. The reals are an open system of numbers, meaning there is a smallest nonnegative, real number, 0, but no largest positive real number nor largest negative real number, which explains the arrowheads graphically on that number line. The hyperreals are similar. They have no largest positive infinite number nor largest negative infinite number (Krakoff, 2015). Omnifinites are different. They have a largest positive infinite number and a largest negative infinite number. These special numbers are absolute (positive) infinity, ∞̶, and absolute negative infinity, ⁻∞̶, respectively, as shown in Figure 1. The whole numbered reals extend finitely to nearly infinity, but infinity is deemed a concept of boundlessness and not a number, since it is nonfinite. For omnifinites and hyperreals, nonfinite numbers may be infinites or infinitesimals as shown in Figure 1. However, there are some differences between omnifinites and hyperreals. With omnifinites, division by zero is fully allowed which is not permitted with hyperreals, nor the reals (Keisler, 2012). This will be discussed in greater detail in the next section. For the hyperreals, the infinites and infinitesimals are specified by using omega, ω, and epsilon, ε, notations, respectively (Keisler, 2012). For simplicity, omnifinites use the original lemniscate notation symbol, ∞, as established by John Wallis (Wallis, 1656) for use in denoting nonfinite numbers such as infinities and infinitesimals. All finite numbers have a hidden term of ∞^0 which equals unity. Nonfinite numbers, that are infinites, have a term, ∞^n, where n is greater than zero. Nonfinite numbers, that are infinitesimals, have the same term, ∞^n, where n is less than zero.

In general, nearly all numbers are well behaved quantities and act as expected. This is true of finite and nonfinite numbers. Whiles most numbers behave alike in terms of operational properties, there are numbers that have unique properties that no other numbers possess. These numbers are zero, 0, and traditional positive and negative infinity, ±∞. For the omnifinites, traditional infinity behaves similarly to any other number as it is just another infinite number in an ocean of infinities. However, absolute infinity, ∞̶, is different. Absolute infinity is the infinity of all infinities. Thus, this number, like zero, has special properties, which differ than all other numbers. Absolute negative infinity behaves in a similar manner as absolute (positive) infinity, but it is a nonpositive or negative number. These three (3) particular numbers, -∞̶, 0, ∞̶, are not the same as other numbers. They are special. Their lack of understanding of how these numbers interacted with other numbers and themselves led to the rise of what was referred to as special forms in mathematics (Cauchy, 1821). Unfortunately, religion and politics in Europe centuries ago led to a type of phobia regarding zero and infinity, where the devil was associated with zero, and God was associated with infinity (Seife, 2000; Aczel, 2014). Due to the rise of European thinking globally which included this uncertainty and phobia of certain arithmetic, mathematical forms became ingrained in the fabric of cultures and mathematical philosophy across the world like a plague or curse that would not dissipate. For over one hundred and fifty years, these special forms have been referred to as indeterminant forms (Moigno, 1840), meaning mathematically these expressions do not have solutions or require mathematical manipulation such as by L'Hopital's rule for solving (L'Hopital, 1696). The truth is our outstanding and lack of knowledge regarding infinity and our inability to properly define it led to these unknown expressions that students have been cautioned on when they are encountered. To this day, generation after generation, century after century, our youth are taught to be cautious of these expressions which have no solutions and have created unwarranted and unneeded uncertainty in the foundation of our understanding of mathematics. Unfortunately, this continues to this very day, and this errorless calculator software tool has been created to shed light and understanding regarding these forms.

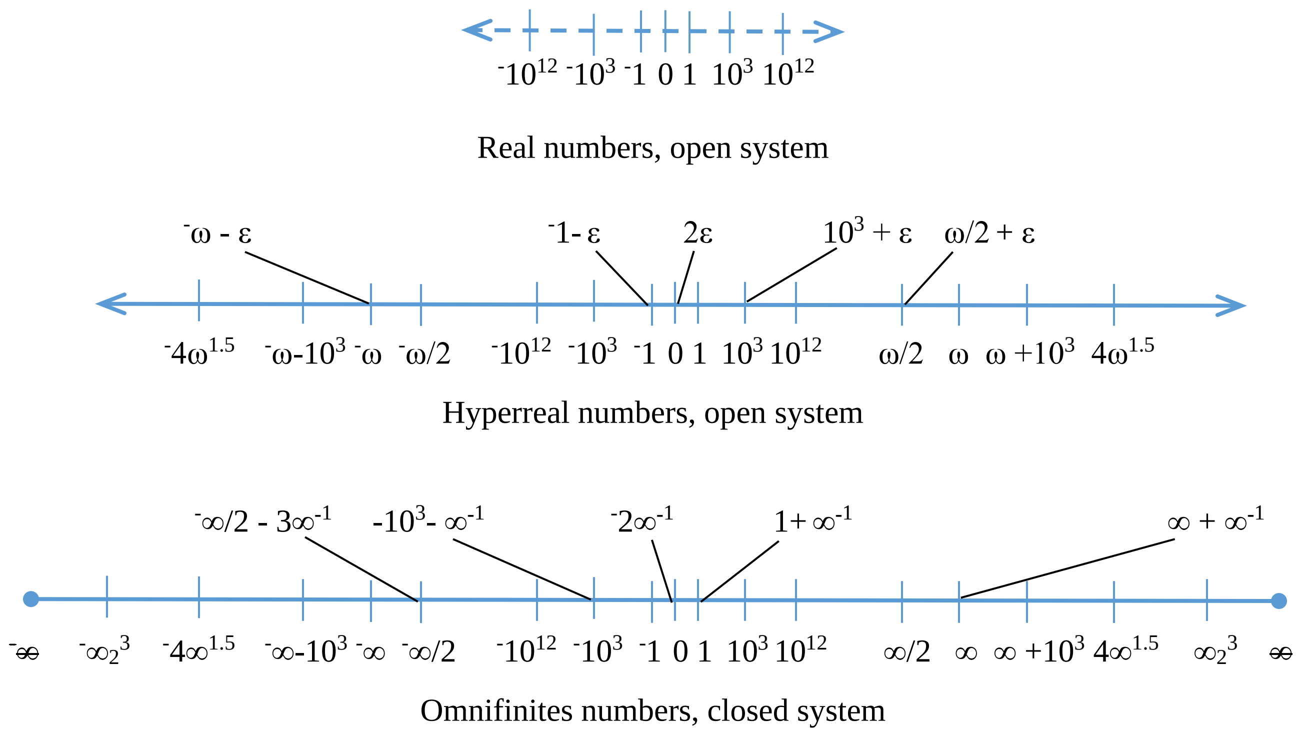

Figure 1. Common number systems in mathematics and the omnifinite number system

Properties of Omnifinites

Zero is a special number; it is an even number and not odd. The signage of zero is neither positive nor negative, and it is the only number with such a property. As such, in terms of omnifinites, specifically absolute infinity, ±⧞, zero may be expressed mathematically by the following two (2) equations.

$$0 = 1 \div ⧞ = {}^{-}1 \div {}^{-}⧞ = ⧞^{-1} \qquad (1)$$

$$0 = {}^{-}1 \div ⧞ = 1 \div {}^{-}⧞ = {}^{-}⧞^{-1} \qquad (2)$$

Zero is an integer and not a prime number. When multiplied by another finite or real number or by a nonfinite number such as an infinitesimal or infinity other than absolute infinity, ±⧞, the result is itself, zero. For omnifinites, division by zero is fully defined and not undefined. Any nonzero number divided by zero is absolute infinity, ±∞̶, where the signage of the resultant is based on the sign of the number in the numerator. In addition, zero divided by zero is also fully defined as follows.

$$0 \div 0 = 1 \qquad (3)$$

For omnifinites, the usage of the lemniscate, ∞, as a symbol for a number does not make the resulting number special, but it does determine if the number is finite or nonfinite based on the value of the exponent of ∞^n as mentioned previously. If the exponent is zero, the number is finite. If the exponent is less than zero, the number is an infinitesimal. If the exponent is greater than zero, the number is an infinite. However, the largest of all infinites, absolute (positive) infinity, ∞̶, and the large negative infinity, absolute negative infinity, $^{-}$∞̶, are special numbers, like zero, and have similar properties. Absolute infinities are even numbers and not odd. The signage of absolute infinity may be positive or negative. For omnifinites, specially zero, absolute infinities, ±∞̶, may be expressed mathematically by the following two (2) equations.

∞̶ = 1 ÷ 0 = 0^{-1} (4)

$^{-}$∞̶ = $^{-}$1 ÷ 0 = $^{-}(0^{-1})$ (5)

Absolute infinities are nonfinite integers. When multiplied by another finite or real number other than zero or by a nonfinite number such as an infinitesimal or an infinite, the resultant is equal to itself, ±∞̶, where the signage of the resultant is based on the sign of the quotients as is the normal practice involving multiplication as shown in the examples in Table 1.

Table 1. Arithmetic examples of multiplication with absolute infinities, ±∞̶

Example†	**Solution**
3 × ∞̶	∞̶
$^{-}$7 × ∞̶	$^{-}$∞̶
2∞ × ∞̶	∞̶
$^{-}9\infty^2$ × ∞̶	$^{-}$∞̶
5 × $^{-}$∞̶	$^{-}$∞̶
$^{-}$6 × $^{-}$∞̶	∞̶
4∞ × $^{-}$∞̶	$^{-}$∞̶
$^{-}8\infty^3$ × $^{-}$∞̶	∞̶

†Order of the numbers not important, same solution results

Division by absolute infinities is fully defined. Any number other than absolute infinities divided by absolute infinities, ±∞̶, is zero as shown in the examples in Table 2.

Absolute infinities divided by absolute infinities are also fully defined as follows.

∞̶ ÷ ∞̶ = 1 (6)

$^{-}$∞̶ ÷ ∞̶ = $^{-}$1 (7)

∞̶ ÷ ($^{-}$∞̶) = $^{-}$1 (8)

$^{-}$∞̶ ÷ ($^{-}$∞̶) = 1 (9)

Table 2. Arithmetic examples of division with absolute infinities, ±⧞

Example†	**Solution**
3 ÷ ⧞	0
$^{-}7$ ÷ ⧞	0
2∞ ÷ ⧞	0
$^{-}9\infty^{2}$ ÷ ⧞	0
5 ÷ $^{-}$⧞	0
$^{-}6$ ÷ ($^{-}$⧞)	0
4∞ ÷ ($^{-}$⧞)	0
$^{-}8\infty^{3}$ ÷ ($^{-}$⧞)	0

†Order of the numbers is important, a different solution results if order is changed

Palmquist's law of number unity states that all numbers have a multiplicative inverse equal to unity as shown in equation (10) as follows.

$$n \times n^{-1} = 1 \qquad (10)$$

Arithmetically, this may be interpreted as any number raised to the exponent of zero or any number divided by itself is also equal to unity as shown in equation (11) as follows,

$$n^{0} = n \div n = 1 \qquad (11)$$

where n is any number, finite or nonfinite including zero and absolute infinities, ±⧞. This applies to all numbers. Since zero is equal to the ratio of 1/⧞ as well as the ratio of $^{-}1/^{-}$⧞, these equations may be multiplied by ⧞ and $^{-}$⧞, respectively, on both sides, which result in the following two (2) equations.

⧞ × 0 = 1 (12)

$^{-}$⧞ × 0 = $^{-}1$ (13)

The seven (7) fundamental indeterminant forms are important in mathematics and needed in the study of functional expressions and limits. In particularly, L'Hopital's rule can be helpful when finding limits of functional expressions or functions that are referred to as indeterminant such as 0/0 and ±⧞/±⧞. However, in terms of arithmetic computations, the resulting numbers are by nature deterministic, and not indeterminant. Arithmetic of numbers cannot yield indeterminant nor undefined numerical expressions in mathematics. They may be nonreal numbers such as ∞, ∞^{2}, or ⧞. In some instances, the solution may not be known or available (as of yet) due the complexity of the numbers and operations being performed, but they are still by nature deterministic. Referring to any simplistic arithmetic computation, especially a basic mathematical expression involving a simple operator and two (2) numbers such as 0, 1, 2, and so forth as indeterminant or undefined, is incorrect. In mathematics, these terms mean that the resulting expressions have no definite or definable value. The implication of saying that anything such as a basic mathematical arithmetic expression has no definitive solution is incorrect and troubling. Yet, this is being taught to our youth every day, and they are also taught not to question it. Words have meaning. Arithmetic computations have solutions which are numbers of some form. The practice of referring to arithmetic expression as indeterminant or undefined needs to end.

With omnifinites, there are no indeterminant arithmetic solutions. All such arithmetic forms have numerical solutions. However, there are special numbers with omnifinites. These numbers have special properties that no other omnifinites have. A couple of these properties are shown in Table 3.

Table 3. Special properties of zero, 0, and absolute infinities, $\pm⧞$

Property	**Zero**	**Absolute Infinity**
Additive self identity	$0 + 0 = 0$	$⧞ + ⧞ = ⧞$ $^{-}⧞ + {}^{-}⧞ = {}^{-}⧞$
Multiplicative self magnitude identity	$\|0\| \times \|0\| = \|0\|$	$\|⧞\| \times \|⧞\| = \|⧞\|$ $\|^{-}⧞\| \times \|^{-}⧞\| = \|^{-}⧞\|$

For the purposes of establishing a direct connection between omnifinites and the hyperreals (Keisler, 2012) as well as other numbers such as the transfinite numbers which include the alephs and ordinals as developed by Prof. Dr. Georg Cantor (Cantor, 1874), the lowest omnifinite whole number that is countably infinite is assigned to be as follows,

$$\infty = \infty_c = \infty_0 = \omega = \aleph_0 = \omega_0 \qquad (14)$$

where c denotes the lowest countable infinite whole number.

The number infinity, ∞, in terms of omnifinites, which is ∞_c, is also equal to ω of the hyperreals. The number infinity, ∞, has also been assigned to be even and equal to Cantor's first countable ordinal, ω_0, and first countable aleph, $\aleph_0$. The lowest omnifinite uncountably infinite whole number is as follows.

$$\infty_{uc} = \infty_1 = \aleph_1 = \omega_1 \qquad (15)$$

The Great Divide, Division by Zero

The great divide refers to division of all nonzero numbers by zero. In mathematics, division by zero is "Undefined." (Alfeld, 1997; Cajori, 1929; Ohm, 1828; Neely, 2000; and Paolilli, 2017). The word "undefined" means an expression which is not assigned an interpretation or value. For more than a millennium, zero has been considered as an actual real number. If one or any other nonzero real number is divided by zero, mathematics has no interpretation, no answer for this. How in the third millennium can there be no mathematical interpretation for a simple arithmetic problem involving two real numbers where division by zero occurs? Why do the axioms of mathematics permit such a significant hole or gap in basic arithmetic mathematics? Perhaps, this is done for mathematical convenience and not mathematical truth. If this is fundamentally true where division by zero has no mathematical answer, then this says something more than just about mathematics and more than just about science. This says something about the very fabric of knowledge and truth. It implies that there are holes or gaps within the fundamental structure of the nature of knowledge and truth. The authors of this work reject this and firmly believe that all fundamental knowledge and truth is fully complete and without holes or gaps though determining or discovering this is challenging and not an easy feat to accomplish. Perhaps, a better answer within the context of the reals, is that division by zero results in a nonreal number which is outside the real number system. Do you think that the most appropriate and acceptable mathematical answer for division by zero is "Undefined."? Are we being mathematical articulate when we specify division by zero as "Undefined."? The authors of this work believe a better and more appropriate answer is warranted and needed.

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The following is an explanation of how division by zero is undefined for the reals. Let us examine the simple function y = 1/x and see how the denominator impacts the arithmetic expression as x approaches zero. This will be examined using two (2) different mathematical perspectives. The first will examine the function of y = 1/x where x approaches zero using a graphical basis in a similar fashion as to how the Greeks would examine such problems. The second perspective will examine this expression from a modern approach using algebra and analysis, specifically mathematical limits. The first perspective may be accomplished by examining the graphs of the functions y = 1/x and y = $^{-}$1/x as shown in Figure 2.

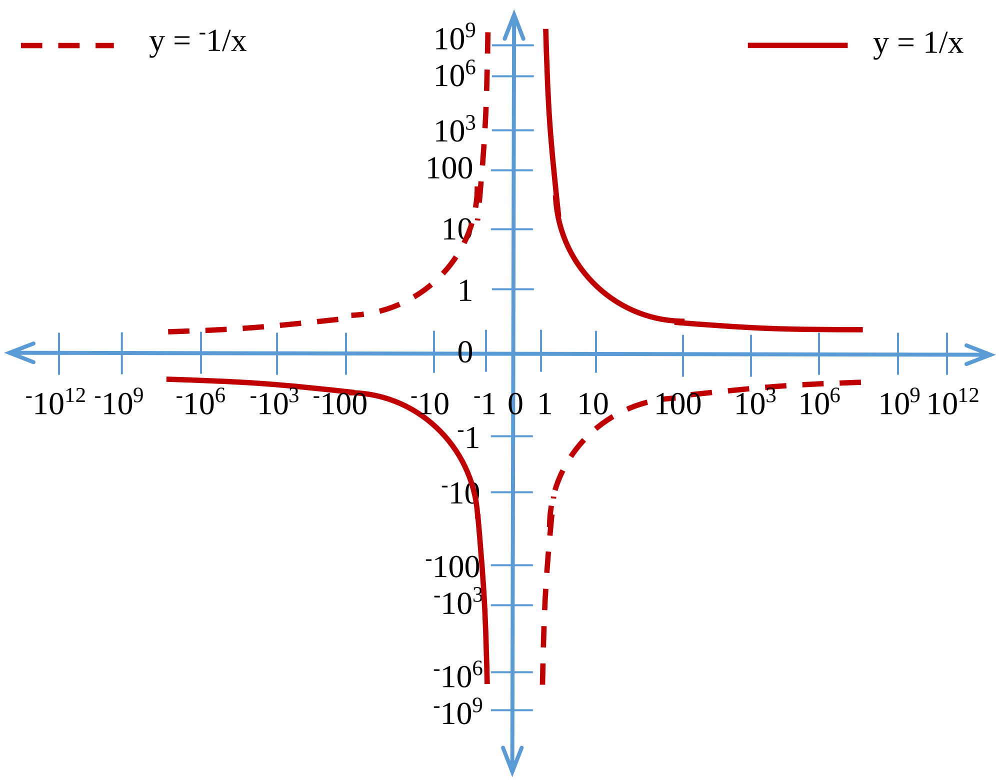

Figure 2. Real number open system with graph of y = 1/x and y = $^{-}$1/x

Looking at the function y = 1/x as x approaches zero from the positive x-axis, the function 1/x approaches a high real number as shown in Figure 2. Likewise, as x approaches zero from the negative x-axis, the function 1/x approaches a high negative number. In addition, looking at the function y = $^{-}$1/x as x approaches zero from the positive x-axis, the function y = $^{-}$1/x approaches a high negative number. Similarly, as x approaches zero from the negative x-axis, the function $^{-}$1/x approaches a high real number. The authors are in full agreement with this result, and this is consistent within the current framework and structure of mathematics with the real number system. Using limit notation, this can be summarized by the following two expressions as follows,

$$\lim_{x\to0^+}\frac{1}{x} = \lim_{x\to0^-}\frac{-1}{x} = n \qquad (16)$$

$$\lim_{x\to0^-}\frac{1}{x} = \lim_{x\to0^+}\frac{-1}{x} = -n \qquad (17)$$

where n is an unspecified high real number.

One could potentially argue that these expressions are approaching infinity and negative infinity, respectively. However, infinity in mathematics is a concept and not a real number. Therefore, putting infinity as the answer is illogical as answers to mathematical problems are not concepts but numerical solutions. As such, n will be specified as a high real number to ensure that only numbers are used or in this case variables representing numbers. From a limit perspective, this implies that 1/0 equals positive or negative high real number depending on the value of x of the domain being examined. Since two reasonable but differing solutions result, n or -n, mathematics currently negates both and labels the associated mathematical arithmetic computation with division by zero as “Undefined.”. In addition, any arithmetic computations with a nonzero numerator where division by zero occurs are also referred to as “Undefined.”, such as $^{-}3/0$, 7/0 and so forth. Seems reasonable. Right…? Well, for functional forms, yes, this is reasonable and correct. However, from an arithmetic mathematical perspective, this is misleading and incorrect. The following explains why.

A different story emerges when the ratio is examined as an arithmetic computation and not as a ratio of functions. Examining the arithmetic computation of this ratio, as the denominator approaches zero with smaller and smaller positive values, the quotient 1/x approaches a high real number. This is identical to the limit approach from the positive x-axis moving towards zero. As the denominator approaches zero from the negative x-axis, the same quotient, 1/x, approaches a negative high real number. Again, this is the same as the limit approach from the negative x-axis moving towards zero. However, this approach is no different than the denominator approaching zero with smaller and smaller positive values while the numerator of the quotient is equal to negative one. Thus, this quotient also approaches a negative high number. Likewise, the quotient $^{-}1/x$ approaches a high number for negative values of the denominator approaching zero.

In evaluating the quotient of 1/0 as a strict arithmetic computation, the numerator must have a purely positive value of one, and the denominator is zero, which is neither positive nor negative. Therefore, the zero in the denominator of 1/0 may not influence the signage of the numerator in any way. In terms of arithmetic, 1/0 may be approximated by using a negative small value for x, say x = $^{-}0.000001$, but only if the numerator cancels this negative value by being equal to negative one as shown below. Thus, the negatives cancel out, and do not impact the signage of the resultant as follows.

$$\frac{-1}{-0.000001} = \frac{1}{0.000001} = 10^6$$

Thus, to properly evaluate the arithmetic computation of 1/0, either of the two (2) following procedures may be used. However, both shall result in the same identical solution which is a high real number.

1. A positive value of one in the numerator of the ratio 1/x and a value in the denominator approaching zero from the positive x-axis results in a high real number, or
2. A negative value of negative one in the numerator of the ratio $^{-}1/x$ and a value in the denominator approaching zero from the negative x-axis also results in a high real number.

Either procedure yields the same exact and unique solution for 1/0 and that is a high real number, as x approaches zero. Figure 3 shows the graphical arithmetical computation of 1/0 and $^{-}1/0$, respectively. The solid purple curves represent 1/0 as shown in quadrants I and II of Figure 3, while the dashed purple curves represent $^{-}1/0$ as shown in quadrants III and IV. Note, the change in quadrants of the solid and dash curves in Figure 3 as compared to Figure 2. This is significant, and the result is correct and logical for all arithmetic computations of the general form of a nonzero and nonabsolute infinity number divided by zero.

From a functional perspective of y = 1/x, the functional value is either a positive or negative high real number depending on the direction from which zero is approached which effects the signage of the quotient being evaluated. In this case, the solution using arithmetic computations appears different than the functional solution which ultimately uses limits. So, why is there a contradiction in the graphs of Figure 2 and 3 depending on whether limits are used versus arithmetic computations? What appears to be a contradiction in mathematics is no contradiction at all. Performing a limit operation and performing an arithmetical operation to determine a result are fundamentally different and have correspondingly differing mathematical meanings. Using limits, the value of the limit of the function or functional form as x approaches the limit in question can often be identified. However, limits do not necessarily give the value of the functional form at the limit itself. In fact, when x equals the limit, the concept of a limit is no longer applicable as per the definition of limits. It is only applicable as you approach the limit. To obtain the value of the expression when x equals the limit, arithmetic computations must be performed.

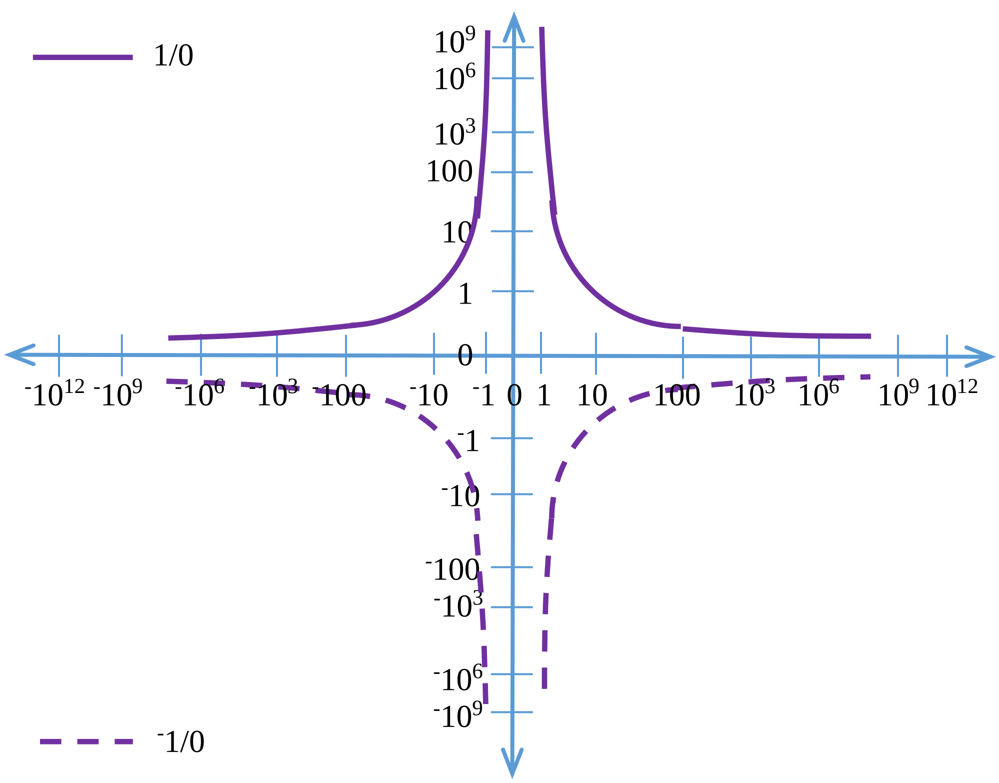

Figure 3. Approximate graphical arithmetical computation of 1/0 and $^{-}1/0$

Arithmetic computations are mathematical operations involving two or more operands. Arithmetic computations are fundamentally different than limits, which are part of mathematical analysis. While limits are important in mathematics, so too are arithmetic computations. Arithmetic computations and functions may be simplified and evaluated at any value that is part of the domain. However, the current practice of mathematics is to disallow the denominator from actually reaching or equaling zero. This practice is not correct.

Simple arithmetic computations involving numbers, such as those described as undefined or indeterminant as discussed herein, all have numerical solutions. Geometry led the Greeks and ultimately all of western European mathematical thinking to negate the significance and importance of zero, and infinity too, perceiving there existence as somehow illegitimate in mathematics. Clearly, philosophy and religion regarding zero and infinity played a part in this as well. Even today, mathematics suffers from these scars since infinity is still not considered as a number despite being utilized routinely in mathematics as a number. Infinity can be used in mathematics as a number. This explains why when John Wallis's first usage of the lemniscate, ∞, was made public, the adoption of this symbol as the number for infinity was so quickly adopted and used throughout the mathematics community. While infinity is not a real number or finite number, it is in fact a number, a nonfinite number. From this, the world's first arithmetic errorless calculator is born.

The First Arithmetic Errorless Calculator

This computer software that functions as a calculator is not without error. However, from a computational perspective, the calculator software algorithm cannot produce an arithmetic error and hence the name and title of this work. All arithmetic errors such as mathematical forms which result in undefined expressions and so forth, which typically and routinely cause mathematical errors, have been removed by defining the underlining arithmetic solution. Other errors are possible such as rounding or exceeding the limits of inputs or outputs but not errors of an arithmetic form. Figure 4 shows a typical phone app calculator and the new errorless calculator.

Typical phone app calculator   Arithmetic errorless calculator

Figure 4. Mathematical arithmetic software

The new software tool is programed to act as a calculator as shown in the schematic diagram in Figure 5. However, the software first scans each input expression looking for division by zero and/or indeterminant forms. Once located, solutions are specified for these expressions which would typically result in a calculator error. The calculator then follows order of operations and completes the arithmetic and specifies a numerical output result.

This is a basic, extended calculator. Basic implies the ability of the calculator to perform simple operations in the correct order based on the inputs. Operations include addition, subtraction, multiplication, division, and exponentiation. In addition, parentheses may be input as part of the arithmetic ordering process when needed. Numbers may be input as positive or negative except for zero. Complex number inputs or complex outputs are not allowed as this is a basic calculator and not scientific. In addition, numbers may be finite or "extend" to nonfinite numbers. Finite numbers are the reals as frequently discussed in mathematics such as 0, 1, e, π, 100, $^{-}1$, $^{-}2$, $^{-}5$, $^{-}109$, and so forth. Nonfinite numbers are infinite or infinitesimal. Infinite numbers may include $\infty^{1/5}$, $\infty^{1/2}$, $\infty/7$, ∞, $\infty/2$, 3∞, 11∞, ∞^{2}, $\infty^{3.2}$, ∞^{∞}, $^{-}\infty^{1/6}$, $^{-}\infty^{2/3}$, $^{-}\infty/3$, $^{-}\infty/11$, $^{-}\infty$, $^{-}4\infty$, $^{-}9\infty$, $^{-}\infty^{4}$, $^{-}\infty^{7.1}$, and $^{-}\infty^{2\infty}$. Infinitesimal numbers may include $\infty^{-1/4}$, $\infty^{-1/2}$, ∞^{-1}, $7\infty^{-1}$, $\infty^{-2.1}$, $\infty^{-5.2}$, $\infty^{-\infty}$, $^{-}\infty^{-2/7}$, $^{-}\infty^{-3/5}$, $^{-}\infty^{-1}$, $^{-}\infty^{-3}$, $^{-}4\infty^{-6.3}$, and $^{-}\infty^{-3\infty}$.

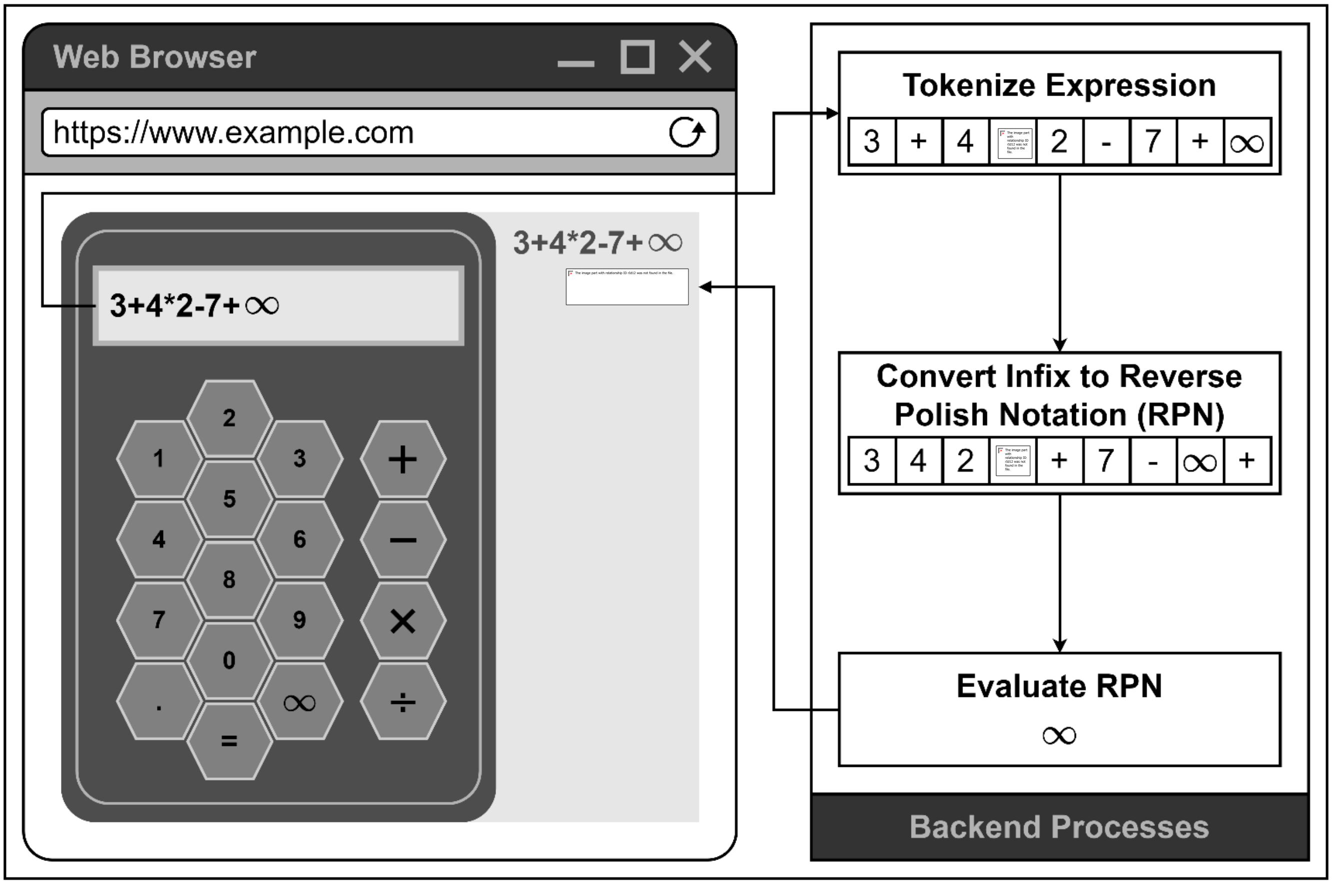

Figure 5. Schematic diagram showing how the errorless calculator performs computations

A new mathematical symbol is introduced referred to as absolute infinity, ⧞. Absolute infinity, ⧞, is the infinity of all infinities. This number may not be exceeded. For example, ⧞ + ⧞ = ⧞ and 5 × ⧞ = ⧞. However, if reduced by subtraction, the resulting value will be zero or absolute infinity itself. For example, ⧞ – ⧞ = 0 and ⧞ – ∞ = ⧞. Likewise, negative absolute infinity, $^{-}$⧞, is the infinity of all negative infinities. For example, $^{-}$⧞ + $^{-}$⧞ = $^{-}$⧞ and 5 × $^{-}$⧞ = $^{-}$⧞. Negative absolute infinity may not be reduced to a number lower than itself. However, if increased by addition, the resulting value will be zero or negative absolute infinity itself. For example, $^{-}$⧞ – $^{-}$⧞ = 0 and $^{-}$⧞ + ∞ = $^{-}$⧞.

Infinity, Absolute Infinity, and Number Format

For this new calculator software tool, which includes reals and nonfinite numbers, the mathematical lemniscate symbol, ∞, behaves differently than is traditionally used. Note, traditional infinity is not regarded as a number but as a mathematical concept of boundlessness. For infinity, ±∞, to be used as an actual number within the omnifinite number system, the mathematical definition must have specificity as discussed in the section, properties of omnifinites. Omnifinite infinity may be compared with traditional infinity as shown in Table 4.

Table 4. Omnifinite infinity compared to traditional infinity

Omnifinite Infinity	Traditional Infinity
$\infty + \infty = 2\infty$	$\infty + \infty = \infty$
$^{-}\infty + {}^{-}\infty = {}^{-}2\infty$	$^{-}\infty + {}^{-}\infty = {}^{-}\infty$
$\infty \times \infty = \infty^2$	$\infty \times \infty = \infty$
$^{-}\infty \times {}^{-}\infty = \infty^2$	$^{-}\infty \times {}^{-}\infty = \infty$

However, absolute infinity, $\not\infty$, arithmetically acts in many cases identical to traditional infinity. However, there are differences. These similarities and differences are shown in Table 5.

Table 5. Omnifinite absolute infinity compared to traditional infinity

Absolute Infinity	Traditional Infinity
$\not\infty + \not\infty = \not\infty$	$\infty + \infty = \infty$
$^{-}\not\infty + {}^{-}\not\infty = {}^{-}\not\infty$	$^{-}\infty + {}^{-}\infty = {}^{-}\infty$
$\not\infty - \not\infty = 0$	$\infty - \infty = \text{NA}$
$^{-}\not\infty - {}^{-}\not\infty = 0$	$^{-}\infty - {}^{-}\infty = \text{NA}$
$\not\infty \times \not\infty = \not\infty$	$\infty \times \infty = \infty$
$^{-}\not\infty \times {}^{-}\not\infty = \not\infty$	$^{-}\infty \times {}^{-}\infty = \infty$
$\not\infty \div \not\infty = 1$	$\infty \div \infty = \text{NA}$
$^{-}\not\infty \div {}^{-}\not\infty = {}^{-}1$	$^{-}\infty \div {}^{-}\infty = \text{NA}$

Note, NA means not allowed. The expression is indeterminant.

Calculator outputs shall return numbers that are arranged in order by size of the type. There are three (3) number types, infinites, finites, and infinitesimals. Positive followed by negative infinites in order of their absolute value in size come first. This is followed by finites in the same manner and then infinitesimals. For example, Table 6 shows the ordering of the outputs based on the following number inputs.

Table 6. Ordering of outputs based on inputs using the errorless calculator tool

Input	Output
$^{-}\infty^{-1} + 78 + 22\infty^2$	$22\infty^2 + 78 - \infty^{-1}$
$13 + 3\infty^{-2} - 7 + 9\infty - 4\infty^{-1}$	$9\infty + 6 - 4\infty^{-1} + 3\infty^{-2}$
$\infty^{-1} + 7 + 5\infty + 2.2\infty^{-1} - 12\infty - 9$	$-7\infty - 2 + 1.1\infty^{-1}$
$141 + 5\infty^{-3} - 4\infty + 61 + 11\infty + 8\infty^4$	$8\infty^4 + 7\infty + 80 + 5\infty^{-3}$

Hands-on Sample Arithmetic Computations

This section provides sample arithmetic computations with solutions which shows the intuitive and simplistic nature of using the errorless calculator. Examples involving addition, subtraction, multiplication, division, exponentiation, and mixed operators are shown in Figures 6 thru 11, respectively.

Input 1	Operation	Input 2	Equals	Output
2.14	+	7.6	=	9.74
5∞	+	14.1∞	=	19.1∞
3.4	+	2∞	=	2∞ + 3.4
9∞	+	∞̶	=	∞̶

Figure 6. Errorless arithmetic with addition

Input 1	Operation	Input 2	Equals	Output
2.14	−	7.6	=	¯5.46
5∞	−	14.1∞	=	¯9.1∞
3.4	−	2∞	=	¯2∞ + 3.4
9∞	−	∞̶	=	¯∞̶

Figure 7. Errorless arithmetic with subtraction

Input 1	Operation	Input 2	Equals	Output
2.14	×	7.6	=	16.264
5∞	×	14.1∞	=	70.5∞^2
3.4	×	2∞	=	6.8∞
9∞	×	∞̶	=	∞̶

Figure 8. Errorless arithmetic with multiplication

Input 1	Operation	Input 2	Equals	Output
2 . 1 4	÷	7 . 6	=	0.28157894736842
5 ∞	÷	1 4 . 1 ∞	=	0.35460992907801
3 . 4	÷	2 ∞	=	$1.7\infty^{-1}$
9 ∞	÷	∞̶	=	0

Figure 9. Errorless arithmetic with division

Input 1	Operation	Input 2	Equals	Output
2 . 1 4	^	7 . 6	=	324.447683302052
5 ∞	^	1 4 . 1 ∞	=	$7{,}169{,}305{,}073^{\infty}\infty^{14.1\infty}$
3 . 4	^	2 ∞	=	11.56^{∞}
9 ∞	^	∞̶	=	∞̶

Figure 10. Errorless arithmetic with exponentiation

Inputs with Mixed Operations	Equals	Output
7 ÷ 2 . 5 + 3 × 4 ∞	=	$12\infty + 2.8$
(2 + ∞) × 3 . 2	=	$3.2\infty + 6.4$
(4 . 1 + 1 . 5 ∞) ^ 2	=	$1.21\infty^{2} + 8.8\infty + 16$
3 ^ 2 × 4 ÷ ∞ + 6	=	$6 + 36\infty^{-1}$

Figure 11. Errorless arithmetic with mixed operations

Conclusions

Our ability to quantify the world around us is important. The new errorless calculator is a software tool which eliminates all arithmetic errors such as those resulting from division by zero or indeterminant forms like zero divided by zero. Arithmetic errorless computations require a mathematical numerically closed system, such as the omnifinite number system as proposed by the authors. An open system, such as the reals or hyperreals, is not robust enough as is and will result in inconsistences or paradoxes which cannot be resolved resulting in mathematical errors. These errors result due to the incomplete fundamental logic of the system.

Special mathematical forms, which were previously thought to have had no solution or be undefined such as division by zero or other forms that are referred to as indeterminant, are now solved arithmetically. The solutions are exact with no approximation. The new software tool as shown herein is relatively simple to use, and the omnifinite number system used by the software has been developed based on simplicity using the lemniscate, ∞, notation symbol for infinities as well as for infinitesimals. As part of the omnifinite number system, two new numbers are introduced. These new numbers are absolute infinity, ⧞, and negative absolute infinity, ⁻⧞. It should be noted that it is possible to create a real number system that is numerically closed where all arithmetic computations may be performed error free. However, modifications to the real number system are required. The omnifinite number system includes all real numbers, finite numbers, as well as all nonfinite numbers, including infinites and infinitesimals, and this is a closed number system. As a result, this software tool is capable of performing error free arithmetic computations, and this will have a significant and global impact on all electronic software that routinely performs mathematical operations on numbers such as calculators.

Handheld calculators have been in existence for more than 50 years now. Mathematical based tools such as calculators, computers, and so forth can be developed that utilize this technology to handle not only real numbers that are finite in nature but also nonfinite numbers which consist of infinities and infinitesimals, as well. Improving our understanding of nonfinite numbers in nature such as in applied fields of study, including physics, chemistry, engineering, and medicine will lead to more innovations in the future.

Acknowledgments

Thank you to the Ritter Family who established the Boyce D. Tate professorship in civil engineering at Western Kentucky University for helping to fund this research. Special thanks to the entire infinity team, including: J. Hall, R. Lake, D. Iglehart, C. Jackson, B. Mazzoni, and T. Thompson.

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Biographies

SHANE M. PALMQUIST is a full professor of structural engineering in the School of Engineering and Applied Sciences at Western Kentucky University (WKU) and is the Boyce D. Tate Professor in civil engineering. Dr. Palmquist is a licensed professional engineer in Kentucky and Florida. Prior to becoming a faculty member at WKU, Dr. Palmquist was a bridge engineer for Lichtenstein Consulting Engineers in Natick, Massachusetts. He earned his B.S. degree in civil engineering from the University of New Hampshire, 1995; an M.S. in structural engineering from the University of Rhode Island, 1996; and his Ph.D. in structures/materials engineering from Tufts University, 2003. His research includes: cable supported structures, high performance materials used in construction, and applied mathematics. Dr. Palmquist may be reached at shane.palmquist@wku.edu.

TIMOTHY J. PENN is a senior completing his bachelor of science degree in mechanical engineering from the School of Engineering and Applied Sciences at Western Kentucky University. Mr. Penn may be reached at timothy.penn769@topper.wku.edu.